Space pre-order and Minus Partial Order for operators on Banach spaces

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Abstract. We extend the definitions of the space pre-order and the minus partial order to the class of bounded linear operators on Banach spaces. Thus, we generalize several results which are well-known for real and complex matrices.

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1. Introduction

For complex matrices A and B of the same order, the space pre-order and the minus partial order, respectively, are defined as follows:

$$A <^s B \Leftrightarrow \mathcal{C}(A) \subseteq \mathcal{C}(B) \text{ and } \mathcal{C}(A^*) \subseteq \mathcal{C}(B^*) \tag{1.1}$$

$$A <^- B \Leftrightarrow A^- A = A^- B \text{ and } AA^- = BA^- \text{ for some } A^- \in \{A^-\}, \tag{1.2}$$

where $\mathcal{C}(A)$ denotes the column space of the matrix A, A^* is the conjugate transpose of A and $\{A^-\}$ denotes the set of all inner generalized inverses of A, i.e. $\{A^-\} = \{G : AGA = A\}$. Notice that in (1.1) the condition $\mathcal{C}(A^*) \subseteq \mathcal{C}(B^*)$ can be replaced by an equivalent condition $\mathcal{N}(B) \subseteq \mathcal{N}(A)$, where $\mathcal{N}(A)$ is the null space of the matrix A. The space pre-order "$<^s$" was defined by Mitra [18], and the minus partial order "$<^-$" by Hartwig [13]. The minus partial order is also called the rank subtractivity order because for matrices A and B of the same order the following equivalence holds, [13]:

$$A <^- B \Leftrightarrow \operatorname{rank}(B - A) = \operatorname{rank}(B) - \operatorname{rank}(A).$$

Our aim is to extend the definitions of the space pre-order and the minus partial order to the class of bounded linear operators on Banach spaces. We generalize a considerably number of results which was proved for real and

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complex matrices. We extend the definition of the minus order to the class of all bounded linear operators which have inner generalized inverses.

Many of the results involving these orders are collected in the monograph [20], see also [1]-[4], [7], [11]-[15], [17], [19], [21], [22], [26]. The proofs in [20] are mostly based on finite dimensional methods. We extend some results to Banach space operators, using operator matrices and infinite dimensional operator theory.

Among other things, generalized inverses are used in solving both matrix and operator equations, see [8], [6]. Also, one may consider the equation $BXC <^- A$. It was considered as matrix equation in [28] and it was consider over a ring in [27].

In [26], Šemrl extended the definition of minus partial order to $\mathcal{B}(H)$, the algebra of bounded linear operators on Hilbert space H.

First we give some notations.

Let X and Y denote arbitrary complex Banach spaces and let $\mathcal{B}(X,Y)$ denote the set of all bounded linear operators from X to Y. Also, $\mathcal{B}(X) = \mathcal{B}(X,X)$. We use $\mathcal{N}(A)$ and $\mathcal{R}(A)$ to denote null-space and range of $A \in \mathcal{B}(X,Y)$, respectively. By a projection we mean a bounded linear operator $P \in \mathcal{B}(X)$ such that $P^2 = P$. Thus a projection is bounded linear idempotent. When P is a projection then $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are closed and $X = \mathcal{R}(P) \oplus \mathcal{N}(P)$.

If $A \in \mathcal{B}(X,Y)$ and there exists some $B \in \mathcal{B}(Y,X)$, such that $ABA = A$ holds, then B is an inner generalized inverse (or in short g-inverse) of A and we say that the operator A is relatively regular.

If $CAC = C$ holds for some $C \in \mathcal{B}(Y,X)$, $C \neq 0$, then C is an outer generalized inverse of A. An operator $D \in \mathcal{B}(Y,X)$ is a reflexive generalized inverse of A, if D is both inner and outer generalized inverse of A.

If C_1, C_2 are inner generalized inverses of A, then C_1AC_2 is a reflexive generalized inverse of A.

The set of all inner (reflexive) generalized inverses of A is denoted by $\{A^-\}$ ($\{A_r^-\}$). Let us denote by $\mathcal{B}_{reg}(X,Y)$ the class of all relatively regular operators from $\mathcal{B}(X,Y)$. Identifying a complex $m \times n$ matrix A with linear operator $A : \mathbb{C}^n \to \mathbb{C}^m$, we conclude that the set of all complex $m \times n$ matrices is equal to $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^m) = \mathcal{B}_{reg}(\mathbb{C}^n, \mathbb{C}^m)$.

The closed subspace $M \subseteq X$ is complemented in X if there exists a closed subspace $N \subseteq X$ such that $X = M \oplus N$. In this case we say that M and N are complementary subspaces in X.

The following lemma is well-known.

Lemma 1.1. *An operator $A \in \mathcal{B}(X,Y)$ is left invertible if and only if A is injective and $\mathcal{R}(A)$ is closed and complemented in Y. An operator $A \in \mathcal{B}(X,Y)$ is right invertible if and only if A is surjective and $\mathcal{N}(A)$ is complemented in X.*

The equivalence of (i) and (ii) in the following lemma is well known [8]. For equivalence of (ii) and (iii) see also [9].

Lemma 1.2. *If $A \in \mathcal{B}(X,Y)$ then the following three conditions are equivalent:*

(i) *A is relatively regular.*

(ii) *$\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, are closed and complemented subspaces of X and Y.*

(iii) *There exists a Banach space Z and the operators $P \in \mathcal{B}(Z,Y)$, $Q \in \mathcal{B}(X,Z)$ such that P is left invertible, Q is right invertible and $A = PQ$.*

Proof. (i) $\Leftrightarrow$ (ii): This is well-known. See [8], 1.1.5 Corollary.
(ii) $\Rightarrow$ (iii): Let $Z = \mathcal{R}(A) \subseteq Y$. Since Y is Banach space and $\mathcal{R}(A)$ is closed, it follows that Z is Banach space. Let us define the operators $Q : X \to Z$ and $P : Z \to Y$ as follows:

$$Qx = Ax, \forall x \in X, \quad \text{and } Pz = z, \forall z \in Z.$$

Then Q is surjective and, by assumption, $\mathcal{N}(Q) = \mathcal{N}(A)$ is closed and complemented in X. Hence, by Lemma 1.1, Q is right invertible. Similarly, P is injective and $\mathcal{R}(P) = Z = R(A)$ is closed and complemented in Y, so P is left invertible. It is clear that $A = PQ$.
(iii) $\Rightarrow$ (ii): Now suppose that $A = PQ$ where P is left invertible and Q is right invertible. It follows that $\mathcal{R}(A) = \mathcal{R}(P)$ and $\mathcal{N}(A) = \mathcal{N}(Q)$. By Lemma 1.1 it follows that $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are closed and complemented subspaces of X and Y respectively. □

When it is the case as in Lemma 1.2 (iii) we say that (P,Q) is the the full-rank decomposition of A, alluding to well-known matrix decomposition.

Notice that if $A \in \mathcal{B}(X,Y)$ is left invertible or right invertible then A is relatively regular.

For this paper, the most important decompositions of spaces are explained further.

Remark 1.3. Suppose that X and Y are Banach spaces such that $X = X_1 \oplus X_2 \oplus X_3$ and $Y = Y_1 \oplus Y_2 \oplus Y_3$, where $X_1, X_2, X_3, X_1 \oplus X_2$ are closed in X and $Y_1, Y_2, Y_3, Y_1 \oplus Y_2$ are closed in Y. The direct sums means that for all $x \in X$ and for all $y \in Y$ there exist unique $x_i \in X_i$ and $y_i \in Y_i$, $i = 1,2,3$, such that $x = x_1 + x_2 + x_3$ and $y = y_1 + y_2 + y_3$.

Let $P : X \to X$, $Q : X_1 \oplus X_2 \to X_1 \oplus X_2$, $R : Y \to Y$ and $S : Y_1 \oplus Y_2 \to Y_1 \oplus Y_2$ be linear idempotents such that $\mathcal{R}(P) = X_1 \oplus X_2$, $\mathcal{N}(P) = X_3$, $\mathcal{R}(Q) = X_1$, $\mathcal{N}(Q) = X_2$, $\mathcal{R}(R) = Y_1 \oplus Y_2$, $\mathcal{N}(R) = Y_3$, $\mathcal{R}(S) = Y_1$ and $\mathcal{N}(S) = Y_2$. Since $X_1 \oplus X_2$ and X_3 are closed and complementary in X, it follows that P is a bounded idempotent, i.e. a projection. This follows from the closed graph theorem. Since X_1 and X_2 are closed and complementary in $X_1 \oplus X_2$, it follows that Q is bounded. Similarly, R and S are bounded idempotents. Hence $I - P$, $I - Q$, $I - R$ and $I - S$ are bounded idempotents too. Of course, operators $P_1 : X \to X_1 \oplus X_2$ and $R_1 : Y \to Y_1 \oplus Y_2$ defined by $P_1 x = Px$, $\forall x \in X$, and $R_1 y = Ry$, $\forall y \in Y$, are bounded too.

Suppose now that $A_{ij} \in \mathcal{B}(X_j, Y_i)$. Finally suppose that mapping $A : X \to Y$ is defined by $Ax = A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + A_{31}x_1 + A_{32}x_2 + A_{33}x_3$, where $x = x_1 + x_2 + x_3$, $x_1 \in X_1$, $x_2 \in X_2$,

$x_3 \in X_3$. It is easy to see that A is linear. It is also bounded. Indeed, for $x = x_1 + x_2 + x_3 \in X_1 \oplus X_2 \oplus X_3$ we have

$$\begin{aligned} \|Ax\| &\leq \|A_{11}x_1\| + \|A_{12}x_2\| + \|A_{13}x_3\| + \|A_{21}x_1\| + \|A_{22}x_2\| + \|A_{23}x_3\| \\ &\quad + \|A_{31}x_1\| + \|A_{32}x_2\| + \|A_{33}x_3\| \leq 3M(\|x_1\| + \|x_2\| + \|x_3\|) \\ &\quad = 3M(\|QP_1x\| + \|(I-Q)P_1x\| + \|(I-P)x\| \\ &\quad \leq 3M(\|Q\|\|P_1\| + \|I-Q\|\|P_1\| + \|I-P\|)\|x\|, \end{aligned}$$

where $M = \max\{\|A_{ij}\| : i,j = 1,2,3\}$. So, $A \in \mathcal{B}(X,Y)$.

Conversely, suppose that $A \in \mathcal{B}(X,Y)$. Let the mappings $A_{ij} : X_j \to Y_i$ are defined by $A_{11}x_1 = SR_1Ax_1$, $A_{12}x_2 = SR_1Ax_2$, $A_{13}x_3 = SR_1Ax_3$, $A_{21}x_1 = (I-S)R_1Ax_1$, $A_{22}x_2 = (I-S)R_1Ax_2$, $A_{23}x_3 = (I-S)R_1Ax_3$, $A_{31}x_1 = (I-R)Ax_1$, $A_{32}x_2 = (I-R)Ax_2$, $A_{33}x_3 = (I-R)Ax_3$, $x_1 \in X_1$, $x_2 \in X_2$, $x_3 \in X_3$. It is easy to see that A_{ij}, $i,j = 1,2,3$, are linear and bounded operators and $Ax = A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + A_{31}x_1 + A_{32}x_2 + A_{33}x_3$, for arbitrary $x = x_1 + x_2 + x_3 \in X_1 \oplus X_2 \oplus X_3 = X$.

In this case we simply write:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

Therefore, if X and Y are Banach spaces such that $X = X_1 \oplus X_2 \oplus X_3$ and $Y = Y_1 \oplus Y_2 \oplus Y_3$, where $X_1, X_2, X_3, X_1 \oplus X_2$ are closed in X and $Y_1, Y_2, Y_3, Y_1 \oplus Y_2$ are closed in Y then A is bounded linear operator if and only if A_{ij}, $i,j \in \{1,2,3\}$, are bounded linear operators on appropriate subspaces. Moreover, the subspaces $X_1 \oplus X_3, X_2 \oplus X_3$ and $Y_1 \oplus Y_3, Y_2 \oplus Y_3$ are also closed in X and Y respectively. To verify this, we consider the following operator:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix},$$

where $I : X_2 \to X_2$ is the identity operator. We conclude from the previous explanation that $A \in \mathcal{B}(X)$, since $I \in \mathcal{B}(X_2)$. Hence A is continuous and therefore $\mathcal{N}(A) = X_1 \oplus X_3$ is closed subspace in X. The closedness of other sums can be proved analogously.

We can define the space pre-order for operators in the same way as for matrices.

Definition 1.4. Let $A, B \in \mathcal{B}(X,Y)$. Then A is said to be below B under the space pre-order, if $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. We denote the space pre-order by '$<^s$', and write $A <^s B$, whenever A is below B under $<^s$.

We define the minus partial order only for relatively regular operators.

Definition 1.5. Let $A, B \in \mathcal{B}_{reg}(X,Y)$ be relatively regular, bounded linear operators from Banach space X to Banach space Y. Then A is said to be below B under the minus partial order if there exists a g-inverse $A^- \in \{A^-\}$

of A such that $AA^- = BA^-$ and $A^-A = A^-B$. We denote the minus order by '$<^-$', and write $A <^- B$, whenever A is below B under $<^-$.

Remark 1.6. Similarly to the case of matrices, the space pre-order is induced by the minus partial order. Suppose that $A, B \in \mathcal{B}_{reg}(X,Y)$ and $A <^- B$, i.e. $AA^- = BA^-$ and $A^-A = A^-B$ for some $A^- \in \{A^-\}$. We know that AA^- is a projection from Y onto $\mathcal{R}(A)$, and $I - A^-A$ is a projection from X onto $\mathcal{N}(A)$. Hence

$$\begin{aligned}\mathcal{R}(A) &= \mathcal{R}(AA^-) = \mathcal{R}(BA^-) \subseteq \mathcal{R}(B)\\ \mathcal{N}(A) &= \mathcal{R}(I - A^-A) = \mathcal{N}(A^-A) = \mathcal{N}(A^-B) \supseteq \mathcal{N}(B).\end{aligned}$$

So,

$$A, B \in \mathcal{B}_{reg}(X,Y) \text{ and } A <^- B \quad \Rightarrow \quad A <^s B.$$

Finally, we recall the well-known Kato theorem (theorem 4.7.5 in [23]).

Theorem 1.7. *Let X, Y be Banach spaces, and let $A \in \mathcal{B}(X,Y)$. If Z is closed subspace in Y, such that $\mathcal{R}(A) \oplus Z$ is closed subspace in Y, then $\mathcal{R}(A)$ is closed in Y.*

2. Space pre-order

The main goal of this section is to give some basic properties and to provide several characterizations of space pre-order. Note that all these properties also hold for real and complex matrices. In our consideration we will need the following lemmas.

Lemma 2.1. *(See also* [10], [5]*) Let X, Y, Z and W be Banach spaces, $A \in \mathcal{B}(X,Y)$, $B \in \mathcal{B}_{reg}(Z,Y)$ and $C \in \mathcal{B}(Z,W)$. Then*

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B) \quad \Leftrightarrow \quad A = BB^-A$, *for each* $B^- \in \{B^-\}$,

(ii) $\mathcal{N}(B) \subseteq \mathcal{N}(C) \quad \Leftrightarrow \quad C = CB^-B$, *for each* $B^- \in \{B^-\}$.

Proof. As B is relatively regular, by Lemma 1.2, it follows that there exists a closed subspace $Z_1 \subseteq Z$, and a closed subspace $Y_1 \subseteq Y$, such that $Z = Z_1 \oplus \mathcal{N}(B)$ and $Y = \mathcal{R}(B) \oplus Y_1$. With regard to these decompositions, it is easy to see that the operator B has the following matrix form:

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Z_1 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ Y_1 \end{bmatrix},$$

where B_1 is invertible. An arbitrary g-inverse B^- of B has the form

$$B^- = \begin{bmatrix} B_1^{-1} & M_2 \\ M_3 & M_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ Y_1 \end{bmatrix} \to \begin{bmatrix} Z_1 \\ \mathcal{N}(B) \end{bmatrix},$$

where M_2, M_3, M_4 are arbitrary bounded linear operators on appropriate subspaces.

(i): If $A = BB^-A$ then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. Suppose that $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. Then operator A is of the form:

$$A = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} : X \to \begin{bmatrix} \mathcal{R}(B) \\ Y_1 \end{bmatrix},$$

where $A_1 x = Ax$, $\forall x \in X$. Now, we obtain that

$$\begin{aligned} BB^-A &= \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1^{-1} & M_2 \\ M_3 & M_4 \end{bmatrix} \begin{bmatrix} A_1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} I & B_1 M_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} = A, \end{aligned}$$

for all $B^- \in \{B^-\}$.
(ii): If $C = CB^-B$ then $\mathcal{N}(B) \subseteq \mathcal{N}(C)$. If $\mathcal{N}(B) \subseteq \mathcal{N}(C)$ it follows that

$$C = \begin{bmatrix} C_1 & 0 \end{bmatrix} : \begin{bmatrix} Z_1 \\ \mathcal{N}(B) \end{bmatrix} \to W,$$

where, $C_1 z = Cz, \forall z \in Z_1$. As before we obtain $CB^-B = C, \forall B^- \in \{B^-\}$. □

Lemma 2.2. *Let $0 \neq x_0 \in X$, $y_0 \in Y$, where X and Y are normed spaces. Then there exists $T \in \mathcal{B}(X,Y)$ such that $Tx_0 = y_0$.*

Proof. From the consequence of the Hahn-Banach theorem it follows that there exists bounded functional $f \in X'$ such that $f(x_0) = 1$. If T is defined by $Tx = f(x)y_0$ then $T \in \mathcal{B}(X,Y)$ and $Tx_0 = y_0$. □

Corollary 2.3. *Let $0 \neq B \in \mathcal{B}(X,Y)$ and $A \in \mathcal{B}(Z,W)$, where X, Y, Z and W are normed spaces. If $ATB = 0$ for all $T \in \mathcal{B}(Y,Z)$ then $A = 0$.*

Proof. Since $B \neq 0$, there exists $x_0 \in X$ such that $Bx_0 = y_0 \neq 0$. Assume to the contrary that there exists some $z_0 \in Z$ such that $Az_0 = w_0 \neq 0$. From Lemma 2.2 it follows that we can find some $T \in \mathcal{B}(Y,Z)$ such that $Ty_0 = z_0$. Hence $ATBx_0 = ATy_0 = Az_0 = w_0 \neq 0$, a contradiction. □

It is well-known that if A and C are non null matrices then AB^-C is invariant under the choices of $B^- \in \{B^-\}$ if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$. This is proved in [24] for complex matrices and in [21] for matrices over arbitrary field. In the next theorem we will show that analogous result is valid for bounded operators on Banach spaces.

Theorem 2.4. *Let $A, B, C \in \mathcal{B}(X,Y)$ be nonnull operators where B is relatively regular. Then the following three conditions are equivalent:*

(i) *AB^-C is invariant under the all choices of $B^- \in \{B^-\}$.*
(ii) *AB_r^-C is invariant under the all choices of $B_r^- \in \{B_r^-\}$.*
(iii) *$\mathcal{R}(C) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$.*

Proof. There exist closed subspaces $X_1 \subseteq X$ and $Y_1 \subseteq Y$ such that $X = X_1 \oplus \mathcal{N}(B)$ and $Y = \mathcal{R}(B) \oplus Y_1$ and with regard to these decompositions we have:

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ Y_1 \end{bmatrix},$$

$$B^- = \begin{bmatrix} B_1^{-1} & K_2 \\ K_3 & K_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ Y_1 \end{bmatrix} \to \begin{bmatrix} X_1 \\ \mathcal{N}(B) \end{bmatrix} \text{ and}$$

$$B_r^- = \begin{bmatrix} B_1^{-1} & M_2 \\ M_3 & M_3 B_1 M_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ Y_1 \end{bmatrix} \to \begin{bmatrix} X_1 \\ \mathcal{N}(B) \end{bmatrix}$$

for some bounded operators K_2, K_3, K_4, M_2, M_3 and invertible operator $B_1 \neq 0$.
(i) $\Rightarrow$ (ii) is trivial.
(ii) $\Rightarrow$ (iii): Suppose that $AB_r^- C$ is invariant under the choice of B_r^-. Assume that A and C have the following matrix forms:

$$A = \begin{bmatrix} A_1 & A_2 \end{bmatrix} : \begin{bmatrix} X_1 \\ \mathcal{N}(B) \end{bmatrix} \to Y \text{ and}$$

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} : X \to \begin{bmatrix} \mathcal{R}(B) \\ Y_1 \end{bmatrix}.$$

We have that

$$AB_r^- C = A_1 B_1^{-1} C_1 + A_1 M_2 C_2 + A_2 M_3 C_1 + A_2 M_3 B_1 M_2 C_2$$

does not depend on M_2 and M_3. If we put $M_2 = M_3 = 0$ it follows $AB_r^- C = A_1 B_1^{-1} C_1$. So if $M_2 = 0$ then $A_2 M_3 C_1 = 0$, $\forall M_3$. Similarly, $A_1 M_2 C_2 = 0$, $\forall M_2$ and so $A_2 M_3 B_1 M_2 C_2 = 0$, $\forall M_2, \forall M_3$. Suppose that $C_2 \neq 0$. From Corollary 2.3 it follows that $A_1 = 0$ and $A_2 M_3 B_1 = 0$, $\forall M_3$. From the same cause $A_2 = 0$ so $A = 0$, a contradiction. Therefore $C_2 = 0$ and since $C \neq 0$ it follows $C_1 \neq 0$. Again from Corollary 2.3 we obtain that $A_2 = 0$. We have just shown that $\mathcal{R}(C) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$.
(iii) $\Rightarrow$ (i): Now suppose $\mathcal{R}(C) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. As in the proof of Lemma 2.1 we conclude that

$$A = \begin{bmatrix} A_1 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ \mathcal{N}(B) \end{bmatrix} \to Y \text{ and}$$

$$C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} : X \to \begin{bmatrix} \mathcal{R}(B) \\ Y_1 \end{bmatrix},$$

for some A_1 and C_1. We have

$$AB^- C = \begin{bmatrix} A_1 & 0 \end{bmatrix} \begin{bmatrix} B_1^{-1} & K_2 \\ K_3 & K_4 \end{bmatrix} \begin{bmatrix} C_1 \\ 0 \end{bmatrix} = A_1 B_1^{-1} C_1,$$

which does not depend on K_2, K_3 and K_4. $\square$

In the next theorem we will give several characterizations of space pre-order.

Theorem 2.5. *Let $A, B \in \mathcal{B}(X, Y)$ where $B \neq 0$ is relatively regular, and let (P, Q) be the full-rank decomposition of B. Then the following six conditions are equivalent:*

(i) $A <^s B$.
(ii) AB^-A *is invariant under the all choices of* $B^- \in \{B^-\}$.
(iii) AB_r^-A *is invariant under the all choices of* $B_r^- \in \{B_r^-\}$.
(iv) $A = BB^-A = AB^-B$, *for all* $B^- \in \{B^-\}$.
(v) $A = BMB$, *for some* $M \in \mathcal{B}(Y, X)$.
(vi) $A = PTQ$, *for some bounded linear operator* T.

Proof. If $A = 0$ then all six conditions are satisfied. Assume that $A \neq 0$. Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follows from Theorem 2.4 and (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) follows from Lemma 2.1.
(i) $\Rightarrow$ (vi): Since $\mathcal{R}(A) \subseteq \mathcal{R}(B) = \mathcal{R}(P)$ and $\mathcal{N}(Q) = \mathcal{N}(B) \subseteq \mathcal{N}(A)$, from Lemma 2.1 it follows $A = PP_l^{-1}A = AQ_r^{-1}Q = PP_l^{-1}AQ_r^{-1}Q = PTQ$, where $T = P_l^{-1}AQ_r^{-1}$ is a bounded linear operator.
(vi) $\Rightarrow$ (i): Since $A = PTQ$ and (P, Q) is the full-rank factorization of B, we obtain $\mathcal{R}(A) \subseteq \mathcal{R}(P) = \mathcal{R}(B)$ and $\mathcal{N}(B) = \mathcal{N}(Q) \subseteq \mathcal{N}(A)$, i.e. $A <^s B$. $\square$

Remark 2.6. If B is relatively regular than it is clear that $A <^s B$ if and only if

$$A = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ Y_1 \end{bmatrix}, \text{ for some } T \in \mathcal{B}(X_1, \mathcal{R}(B)).$$

Also $0 <^s A$ for all A and $A <^s 0$ if and only if $A = 0$.

3. Minus order

Now, we investigate the minus order for relatively regular operators on Banach spaces. There are many characterizations of minus partial order. In matrix case the equivalences:

$$\begin{aligned} A <^- B &\Leftrightarrow \operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(B - A), \\ B = A \oplus (B - A) &\Leftrightarrow \{B^-\} \subseteq \{A^-\}, \\ A <^- B &\Leftrightarrow \{B^-\} \subseteq \{A^-\}, \\ \{B^-\} \subseteq \{A^-\} &\Leftrightarrow \{B_r^-\} \subseteq \{A^-\}, \end{aligned}$$

are proved in [13], [16], [19], [25], respectively.
It is proved in [14] that $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(B - A)$ if and only if there exist unitary matrices U and V such that

$$A = U \begin{bmatrix} D_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^* \text{ and } B = U \begin{bmatrix} D_a + RD_{b-a}S & RD_{b-a} & 0 \\ D_{b-a}S & D_{b-a} & 0 \\ 0 & 0 & 0 \end{bmatrix} V^*,$$

where D_a and D_{b-a} are diagonal matrices of orders $a \times a$ and $(b-a) \times (b-a)$ with positive diagonal elements and $a = \operatorname{rank}(A)$, $b = \operatorname{rank}(B)$.
Also, it is proved in [20] that $A <^- B$ if and only if there exist non-singular matrices R and S such that

$$A = R\operatorname{diag}(I_a, 0, 0)S \text{ and } B = R\operatorname{diag}(I_a, I_{b-a}, 0)S,$$

where I_a and I_{b-a} are identity matrices.
The following theorem is the main result of this paper. It generalizes the above equivalences to the class $\mathcal{B}_{reg}(X,Y)$.

Theorem 3.1. *Suppose that $A, B \in \mathcal{B}_{reg}(X,Y)$, where X and Y are Banach spaces. Then the following are equivalent:*

(i) $A <^- B$
(ii) $\mathcal{R}(B) = \mathcal{R}(A) \oplus \mathcal{R}(B-A)$
(iii) *There exist decompositions $X = X_1 \oplus X_2 \oplus \mathcal{N}(B)$ and $Y = \mathcal{R}(A) \oplus \mathcal{R}(B-A) \oplus Y_1$ for some closed subspaces $X_1, X_2 \subseteq X$, $Y_1 \subseteq Y$ such that $X_1 \oplus X_2$, $\mathcal{R}(B-A)$ and $\mathcal{R}(A) \oplus \mathcal{R}(B-A)$ are closed and there exist invertible bounded operators $C_1 \in \mathcal{B}(X_1, \mathcal{R}(A))$ and $C_2 \in \mathcal{B}(X_2, \mathcal{R}(B-A))$ such that*

$$A = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ Y_1 \end{bmatrix} \quad \textit{and}$$

$$B = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ Y_1 \end{bmatrix}.$$

(iv) $\{B^-\} \subseteq \{A^-\}$.
(v) $\{B_r^-\} \subseteq \{A^-\}$.

Proof. We may assume that $B \neq 0$ because if $B = 0$ then each of conditions (i)-(v) is equivalent to condition $A = 0$.
(i) $\Rightarrow$ (ii): Since B is relatively regular, it follows by Lemma 1.2 that $B = PQ$ where $P \in \mathcal{B}(\mathcal{R}(B), Y)$ is left invertible, $Py = y, \forall y \in \mathcal{R}(B)$ and $Q \in \mathcal{B}(X, \mathcal{R}(B))$ is right invertible, $Qx = Bx, \forall x \in X$. By Remark 1.6 it follows $A <^s B$, and hence by Theorem 2.5 we obtain that $A = PTQ$ for some $T \in \mathcal{B}(\mathcal{R}(B))$. By the definition of the minus partial order there exists $A^- \in \{A^-\}$ such that $AA^- = BA^-$ and $A^-A = A^-B$. Let $G = A^-AA^-$. Then $G \in \{A_r^-\}$, $AG = BG$ and $GA = GB$. Hence $PTQG = PQG$ so $TQG = QG$ since P is left invertible. Next, $AGA = A$, that is, $PTQGPTQ = PTQ$. Thus $TQGPT = T$ and therefore $QGPT = T$. Now from $GAG = G$ we obtain $T^2 = (QGPT)(QGPT) = QGAGPT = QGPT = T$. So $A = PTQ$ where $T \in \mathcal{B}(\mathcal{R}(B))$ is a projection. It follows that $\mathcal{R}(B) = \mathcal{R}(T) \oplus \mathcal{N}(T)$. Since Q is right invertible and $Py = y$, $\forall y \in \mathcal{R}(B)$, it follows that $\mathcal{R}(A) = \mathcal{R}(PTQ) = \mathcal{R}(PT) = \mathcal{R}(T)$. Similarly, $\mathcal{R}(B-A) = \mathcal{R}(PQ - PTQ) = \mathcal{R}(P(I-T)Q) = \mathcal{R}(P(I-T)) = \mathcal{R}(I-T) = \mathcal{N}(T)$. Thus we have proved $\mathcal{R}(B) = \mathcal{R}(A) \oplus \mathcal{R}(B-A)$.
(ii) $\Rightarrow$ (iii): Suppose that $\mathcal{R}(B) = \mathcal{R}(A) \oplus \mathcal{R}(B-A)$. Since $\mathcal{N}(B)$ and $\mathcal{R}(B)$, respectively, are closed and complemented subspaces of X and Y it follows that there exist closed subspaces $X_3 \subseteq X$ and $Y_1 \subseteq Y$ such that $X = X_3 \oplus \mathcal{N}(B)$ and $Y = \mathcal{R}(B) \oplus Y_1$. Then operator B has the following matrix form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_3 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ Y_1 \end{bmatrix},$$

where B_1 is invertible. Let $X_1 = B_1^{-1}(\mathcal{R}(A))$ and $X_2 = B_1^{-1}(\mathcal{R}(B-A))$. Since $\mathcal{R}(B) = \mathcal{R}(A) \oplus \mathcal{R}(B-A)$ and since $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, from Kato Theorem 1.7 we conclude that $\mathcal{R}(B-A)$ is closed too. Since B_1 is bounded (equivalently continuous), it follows that X_1 and X_2 are closed. Since B_1 is invertible, we deduce that $X_3 = X_1 \oplus X_2$. Suppose further that $x \in \mathcal{N}(B)$. Since $0 = Bx = Ax + (B-A)x \in \mathcal{R}(A) \oplus \mathcal{R}(B-A)$ it follows that $Ax = 0 = (B-A)x$, so $\mathcal{N}(B) \subseteq \mathcal{N}(A)$.
It follows from the above discussion that A and B have the following matrix forms:

$$A = \begin{bmatrix} K & L & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ Y_1 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ Y_1 \end{bmatrix},$$

for some bounded operators K, L and some invertible bounded operators C_1, C_2 defined on appropriate subspaces.
Let us show that $K = C_1$ and $L = 0$. Let $x \in X_1$. Then $Bx = C_1x + 0 \in \mathcal{R}(A) \oplus \mathcal{R}(B-A)$. On the other hand, $Bx = Ax + (B-A)x = Kx + (B-A)x \in \mathcal{R}(A) \oplus \mathcal{R}(B-A)$. We conclude that $Kx = C_1x, \forall x \in X_1$, that is, $K = C_1$. Similarly, for $x \in X_2$, we have $Bx = 0 + C_2x \in \mathcal{R}(A) \oplus \mathcal{R}(B-A)$. On the other hand $Bx = Ax + (B-A)x = Lx + (B-A)x \in \mathcal{R}(A) \oplus \mathcal{R}(B-A)$, so $Lx = 0, \forall x \in X_2$, i.e. $L = 0$.
(iii) $\Rightarrow$ (iv): Suppose that (iii) holds. Then an arbitrary $G \in \{B^-\}$ is of the form

$$G = \begin{bmatrix} C_1^{-1} & 0 & G_{13} \\ 0 & C_2^{-1} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ Y_1 \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \\ \mathcal{N}(B) \end{bmatrix},$$

for some operators $G_{13}, G_{23}, G_{31}, G_{32}, G_{33}$. It is easy to verify that $AGA = A$, that is, $\{B^-\} \subseteq \{A^-\}$.
(iv) $\Rightarrow$ (v) is trivial.
(v) $\Rightarrow$ (i): Since $AB_r^-A = A, \forall B_r^- \in \{B_r^-\}$, from Theorem 2.5 (iii) $\Leftrightarrow$ (iv) it follows that $A = BB^-A = AB^-B, \forall B^- \in \{B^-\}$. For arbitrary $B^- \in \{B^-\}$, $G = B^-BB^- \in \{B_r^-\}$ so $A = AGA = BGA = AGB$. Let $F = GAG$. Then $AFA = AGAGA = A$, i.e. $F \in \{A^-\}$. Also, $AF = AGAG = AG = BGAG = BF$ and $FA = GAGA = GA = GAGB = FB$. Hence $A <^- B$. $\square$

When it is the case as in Theorem 3.1, we say that the decompositions $X = X_1 \oplus X_2 \oplus \mathcal{N}(B)$ and $Y = \mathcal{R}(A) \oplus \mathcal{R}(B-A) \oplus Y_1$ are standard decompositions of X and Y. We will see that representation of operators with respect to these decompositions, in the case when $A <^- B$, is crucial in proving most of the following theorems.

Corollary 3.2. *Let the operator $B \in \mathcal{B}_{reg}(X,Y)$ has the full-rank decomposition $B = PQ$. Then the class of all operators $A \in \mathcal{B}_{reg}(X,Y)$ such that $A <^- B$ is given by $\{PTQ : T$ is a projection$\}$.*

Proof. Suppose that $A <^- B$. As in the proof (i) $\Rightarrow$ (ii) of Theorem 3.1 we obtain $A = PTQ$ for some projection T. If $A = PTQ$ where T is projection then for $G = Q_r^{-1}TP_l^{-1}$ we have $AGA = A$, $AG = PTP_l^{-1} = BG$, and $GA = Q_r^{-1}TQ = GB$ so $A <^- B$. □

Theorem 3.3. *The minus partial order is a partial order on $\mathcal{B}_{reg}(X,Y)$.*

Proof. From Theorem 3.1 (iv), reflexivity and transitivity holds trivially. If $A <^- B$ and $B <^- A$, where $A, B \in \mathcal{B}_{reg}(X,Y)$ then, from Theorem 3.1 (ii), it follows $\mathcal{R}(B) = \mathcal{R}(A) \oplus \mathcal{R}(B - A)$ and $\mathcal{R}(A) = \mathcal{R}(B) \oplus \mathcal{R}(A - B)$. Hence $\mathcal{R}(B - A) = \{0\}$, that is $A = B$ and '$<^-$' is a partial order on $\mathcal{B}_{reg}(X,Y)$. □

Corollary 3.4. *Let $A, B \in \mathcal{B}(X,Y)$. If $\{A^-\} = \{B^-\} \neq \emptyset$ then $A = B$.*

Remark 3.5. Let $A, B \in \mathcal{B}_{reg}(X,Y)$ and $A <^- B$. Then $B - A$ is relatively regular.
Indeed, from Theorem 3.1 it follows that

$$B - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B - A) \\ Y_1 \end{bmatrix},$$

where C_2 is invertible, so we conclude that $B - A$ is relatively regular.

Remark 3.6. If T and S are invertible operators and $A, B \in \mathcal{B}_{reg}(X,Y)$ then $A <^- B$ if and only if $TAS <^- TBS$. If $A, B \in \mathcal{B}_{reg}(X,Y)$ where A is left or right invertible and if $A <^- B$ then $A = B$.
Since $\{(TXS)^-\} = \{S^{-1}X^-T^{-1}\}$, the first assertion follows from Theorem 3.1 (i) $\Leftrightarrow$ (iv).
If A_r^{-1} is a right inverse of A then we have the following sequence of implications: $A <^- B \Rightarrow A = AB^-A = BB^-A = AB^-B \Rightarrow I = AA_r^{-1} = AB^-AA_r^{-1} = AB^- \Rightarrow A = AB^-B = IB = B$. The case when A is left invertible is similar.

In the following theorem we will give a number of equivalent conditions for minus partial order. The conditions analogous to (i)-(x) are also equivalent in any regular semigroup, [22]. The equivalence of (i) and (xiii) is also valid in regular ring, [13]. It is proved in [20] Theorem 3.3.16 that for real matrices A and B, $A <^- B$ if and only if

$$\operatorname{rank}(B - A) = \operatorname{rank}((I - AA^-)B) = \operatorname{rank}(B(I - A^-A)), \forall A^- \in \{A^-\}.$$

In the infinite dimensional case we can not use rank, so we use the image and the null-space of a given operator. Notice that our conditions in (xiv) and (xv) are weaker than above condition.

Theorem 3.7. *Let $A, B \in \mathcal{B}_{reg}(X,Y)$. Then the following statements are equivalent:*

(i) $A <^- B$;
(ii) $B - A <^- B$;
(iii) $AA_r^- = BA_r^-$ *and* $A_r^- A = A_r^- B$ *for some* $A_r^- \in \{A_r^-\}$;
(iv) $A = BA_r^- A = AA_r^- B$ *for some* $A_r^- \in \{A_r^-\}$;
(v) $A = AB^- B = BB^- A = AB^- A$ *for all* $B^- \in \{B^-\}$;
(vi) $A <^s B$ *and* $\{A^-\} \cap \{B^-\} \neq \emptyset$;
(vii) $A = AB^- B = BB^- A = AB^- A$ *for some* $B^- \in \{B^-\}$;
(viii) $A = PB = BQ$ *for some projections* $P \in \mathcal{B}(Y)$ *and* $Q \in \mathcal{B}(X)$;
(ix) $A = PB = BM$ *for some projection* $P \in \mathcal{B}(Y)$ *and some operator* $M \in \mathcal{B}(X)$;
(x) $A = KB = BM$ *and* $KA = A$ *for some operators* $K \in \mathcal{B}(Y)$ *and* $M \in \mathcal{B}(X)$;
(xi) $A^- A = A^- B$ *and* $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ *for some* $A^- \in \{A^-\}$;
(xii) $AA^- = BA^-$ *and* $\mathcal{N}(B) \subseteq \mathcal{N}(A)$ *for some* $A^- \in \{A^-\}$;
(xiii) $B = A + (I - AA^-)W(I - A^- A)$ *for some* $A^- \in \{A^-\}$ *and some* $W \in \mathcal{B}(X, Y)$;
(xiv) $\mathcal{N}((I - AA^-)B) \subseteq \mathcal{N}(B - A)$ *and* $\mathcal{R}(A) \subset \mathcal{R}(B)$ *for all* $A^- \in \{A^-\}$;
(xv) $\mathcal{R}(B - A) \subseteq \mathcal{R}(B(I - A^- A))$ *and* $\mathcal{N}(B) \subset \mathcal{N}(A)$ *for all* $A^- \in \{A^-\}$.

Proof. Some equivalences can be proved as in the case of regular semigroup, [22]. It is proved here for completeness.
(i) $\Leftrightarrow$ (ii) follows from equivalence of (i) and (ii) of Theorem 3.1.
(i) $\Rightarrow$ (iii): There exists $A^- \in \{A^-\}$ such that $AA^- = BA^-$ and $A^- A = A^- B$. For $A_r^- = A^- AA^-$ (iii) is satisfied.
(iii) $\Rightarrow$ (iv) is trivial.
(iv) $\Rightarrow$ (i): For any $B^- \in \{B^-\}$ it holds $AB^- A = AA_r^- BB^- BA_r^- A = AA_r^- BA_r^- A = AA_r^- A = A$. The result follows from equivalence of (i) and (iv) of Theorem 3.1.
(i) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) follows from Theorems 2.5 and 3.1.
(vii) $\Rightarrow$ (viii): Let $P = AB^-$ and $Q = B^- A$. Then P and Q are projections and $A = PB = BQ$.
(viii) $\Rightarrow$ (ix) $\Rightarrow$ (x) is trivial.
(x) $\Rightarrow$ (i): For any $B^- \in \{B^-\}$ it holds $AB^- A = KBB^- BM = KBM = KA = A$.
(i) $\Rightarrow$ (xi) is trivial.
(xi) $\Rightarrow$ (i): $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ so $A = BB^- A$, $\forall B^- \in \{B^-\}$ and hence $AB^- A = AA^- AB^- A = AA^- BB^- BB^- A = AA^- BB^- A = AA^- A = A$.
(i) $\Leftrightarrow$ (xii): This part is similar to the proof of (i) $\Leftrightarrow$ (xi).
(i) $\Rightarrow$ (xiii): Suppose that $A <^- B$, $B^- \in \{B^-\}$ and $G = B^- BB^-$. Then $G \in \{B_r^-\} \subseteq \{A^-\}$. From (i) $\Rightarrow$ (v) , we have $A = BGA = AGB = AGA$. Let $A^- = G$ and $W = B$. Than it is easy to show that $A + (I - AA^-)W(I - A^- A) = B$.
(xiii) $\Rightarrow$ (i): Let $G = A^- AA^-$. Then $G \in \{A^-\}$ and from assumption it follows that $BG = AG$ and $GB = GA$.

(i) $\Rightarrow$ (xiv): From Theorem 3.1 it follows that

$$A^- = \begin{bmatrix} C_1^{-1} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ Y_1 \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \\ \mathcal{N}(B) \end{bmatrix}$$

and hence

$$(I - AA^-)B = \begin{bmatrix} 0 & -C_1G_{12}C_2 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ Y_1 \end{bmatrix}$$

For $x \in \mathcal{N}((I - AA^-)B)$, $x = x_1 + x_2 + x_3 \in X_1 \oplus X_2 \oplus \mathcal{N}(B)$ we have $0 = -C_1G_{12}C_2x_2 + C_2x_2 \in \mathcal{R}(A) \oplus \mathcal{R}(B-A)$ which is equivalent to $C_2x_2 = 0$ i.e. $x_2 = 0$. Hence $\mathcal{N}((I - AA^-)B) = X_1 \oplus \mathcal{N}(B) = \mathcal{N}(B - A)$. Of course, $A <^- B \Rightarrow \mathcal{R}(A) \subseteq \mathcal{R}(B)$.
(xiv) $\Rightarrow$ (i): From $\mathcal{N}((I-AA^-)B) \subseteq \mathcal{N}(B-A)$ it follows $\mathcal{N}(B) \subseteq \mathcal{N}(B-A)$, so $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. Therefore $A <^s B$ and hence

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ Y_1 \end{bmatrix},$$
$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ Y_1 \end{bmatrix} \text{ and}$$
$$A^- = \begin{bmatrix} A_1^- & G_2 \\ G_3 & G_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ Y_1 \end{bmatrix} \to \begin{bmatrix} X_1 \\ \mathcal{N}(B) \end{bmatrix},$$

where X_1 and Y_1 are closed subspaces, B_1 is invertible and $A_1^- \in \{A_1^-\}$ (A_1 is relatively regular because A is relatively regular). Now, $\mathcal{N}((I - A_1A_1^-)B_1) = \mathcal{N}((I - AA^-)B) \subseteq \mathcal{N}(B - A) = \mathcal{N}(B_1 - A_1) = \mathcal{N}((I - A_1B_1^{-1})B_1)$. Since B_1 is invertible it follows that $\mathcal{N}(I - A_1A_1^-) \subseteq \mathcal{N}(I - A_1B_1^{-1})$, that is $\mathcal{R}(A_1) \subseteq \mathcal{N}(I - A_1B_1^{-1})$. Thus we have proved that $(I - A_1B_1^{-1})A_1 = 0$. Therefore $B_1^{-1} \in \{A_1^-\}$ and hence $\{B^-\} \subseteq \{A^-\}$, i.e., $A <^- B$.
(i) $\Leftrightarrow$ (xv): This part can be proved in a similar way as (i) $\Leftrightarrow$ (xiv). $\square$

Let

$$\{A^-\}_B = \{G \in \{A^-\} : AG = BG, GA = GB\} \text{ and}$$
$$\{A_r^-\}_B = \{G \in \{A_r^-\} : AG = BG, GA = GB\}.$$

In the next theorem we obtain explicit representations of $\{A^-\}_B$ and $\{A_r^-\}_B$ (for matrix case see [17] and [18]).

Theorem 3.8. *Let $A, B \in \mathcal{B}_{reg}(X, Y)$ such that $A <^- B$. Then*

(i) $\{A^-\}_B = \{B^- - B^-(B - A)B^- : B^- \in \{B^-\}\}$
(ii) $\{A_r^-\}_B = \{B^-AB^- : B^- \in \{B^-\}\} = \{B_r^-AB_r^- : B_r^- \in \{B_r^-\}\}$.

Proof. (i): Let us denote the right-hand side of (i) by R. Since $A <^- B$ we have the following representations with respect to standard decompositions given in Theorem 3.1:

$$A = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B^- = \begin{bmatrix} C_1^{-1} & 0 & H_{13} \\ 0 & C_2^{-1} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix},$$

where $H_{13}, H_{23}, H_{31}, H_{32}, H_{33}$ are arbitrary bounded operators. It follows that $G \in R$ if and only if

$$G = B^- - B^-(B-A)B^- = \begin{bmatrix} C_1^{-1} & 0 & H_{13} \\ 0 & 0 & 0 \\ H_{31} & 0 & K \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ Y_1 \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \\ \mathcal{N}(B) \end{bmatrix},$$

for a particular choice of H_{13}, H_{31} and K ($K = H_{33} - H_{32}C_2H_{23}$). Now, it is easy to show that $AGA = A$, $AG = BG$ and $GA = GB$, i.e. $G \in \{A^-\}_B$. Assume now that $G \in \{A^-\}_B$. Then

$$G = \begin{bmatrix} C_1^{-1} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix},$$

for some operators G_{ij}. From $AG = BG$ and $GA = GB$ we obtain $G_{12} = G_{21} = G_{22} = G_{23} = G_{32} = 0$. It is easy to show that $G = B^- - B^-(B - A)B^- \in R$ where

$$B^- = \begin{bmatrix} C_1^{-1} & 0 & G_{13} \\ 0 & C_2^{-1} & 0 \\ G_{31} & 0 & G_{33} \end{bmatrix}.$$

(ii): The proof of (ii) may be obtain in a similar way. We obtain that the set $\{A_r^-\}_B$ is given by:

$$\begin{bmatrix} C_1^{-1} & 0 & G_{13} \\ 0 & 0 & 0 \\ G_{31} & 0 & G_{31}C_1G_{13} \end{bmatrix},$$

where G_{13} and G_{31} are arbitrary. □

Let $A <^- B$, where $A, B \in \mathbb{C}^{n\times n}$ are complex matrices, $a = \operatorname{rank}(A) < \operatorname{rank}(B) = b$ and let $c_1, c_2 \in \mathbb{C}$, $c_2 \neq 0$, $c_1 + c_2 \neq 0$. In [29] authors proved that $c_1A + c_2B$ is nonsingular if and only if B is nonsingular. Furthermore, they proved that in this case the following formula holds:

$$(c_1A+c_2B)^{-1} = (c_1+c_2)^{-1}B^{-1} + (c_2^{-1} - (c_1+c_2)^{-1})[(0\oplus I_{n-a})B(0\oplus I_{n-a})]^\dagger,$$

where $0 \oplus I_{n-a} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-a} \end{bmatrix}$ and $(\cdot)^\dagger$ is the Moore-Penrose inverse of $(\cdot)$.

The next theorem shows that the same result is valid when $A, B \in \mathcal{B}_{reg}(X, Y)$. We obtain the more convenient formula for $(c_1A + c_2B)^{-1}$.

Theorem 3.9. *Let $A <^- B$ where $A, B \in \mathcal{B}_{reg}(X, Y)$ and let $c_1, c_2 \in \mathbb{C}$, $c_2 \neq 0$, $c_1 + c_2 \neq 0$. Then $c_1A + c_2B$ is invertible if and only if B is invertible.*

Furthermore,

$$(c_1A + c_2B)^{-1} = c_2^{-1}B^{-1} + ((c_1 + c_2)^{-1} - c_2^{-1})B^{-1}AB^{-1} \tag{3.1}$$
$$= c_2^{-1}B^{-1} + ((c_1 + c_2)^{-1} - c_2^{-1})A^-,$$

where $A^- \in \{A^-\}_B$.

Proof. Since $A <^- B$ then, according to Theorem 3.1, we have the following representations with respect to standard decompositions $X = X_1 \oplus X_2 \oplus \mathcal{N}(B)$, $Y = \mathcal{R}(A) \oplus \mathcal{R}(B - A) \oplus Y_1$:

$$A = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$c_1A + c_2B = \begin{bmatrix} (c_1 + c_1)C_1 & 0 & 0 \\ 0 & c_2C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where C_1 and C_2 are invertible operators. Since $c_2 \neq 0$, $c_1 + c_2 \neq 0$, it is now clear that B is invertible if and only if $c_1A + c_2B$ is invertible if and only if $\mathcal{N}(B) = \{0\}$ and $Y_1 = \{0\}$. In this case $X = X_1 \oplus X_2$, $Y = \mathcal{R}(A) \oplus \mathcal{R}(B - A)$ and with respect to these decompositions we have:

$$B^{-1} = \begin{bmatrix} C_1^{-1} & 0 \\ 0 & C_2^{-1} \end{bmatrix} \text{ and } (c_1A+c_2B)^{-1} = \begin{bmatrix} (c_1 + c_2)^{-1}C_1^{-1} & 0 \\ 0 & c_2^{-1}C_2^{-1} \end{bmatrix},$$

so the formula (3.1) can be easily checked.

Since B is invertible and $A <^- B$ it follows from Theorem 3.8 that $\{A^-\}_B = \{B^{-1}AB^{-1}\}$. □

Theorem 3.10. (see [20] Theorem 3.5.6) *Let* $A, B \in \mathcal{B}_{reg}(X, Y)$ *such that* $A <^- B$. *Then*

(i) *For any* $A^- \in \{A^-\}_B$ *there exists* $B^- \in \{B^-\}$ *such that* $B^-A = A^-A$ *and* $AB^- = AA^-$.
(ii) *For any* $B^- \in \{B^-\}$ *there exists* $A^- \in \{A^-\}_B$ *such that* $AA^- = AB^-$ *and* $A^-A = B^-A$.

Proof. (i): From Theorem 3.8 we conclude that $A^- \in \{A^-\}_B$ has the following form with respect to standard decompositions of X and Y:

$$A^- = \begin{bmatrix} C_1^{-1} & 0 & H_{13} \\ 0 & 0 & 0 \\ H_{31} & 0 & H_{33} \end{bmatrix},$$

for some operators H_{13}, H_{31}, H_{33}. It is easy to show that $B^-A = A^-A$ and $AB^- = AA^-$ where

$$B^- = \begin{bmatrix} C_1^{-1} & 0 & H_{13} \\ 0 & C_2^{-1} & G_{23} \\ H_{31} & G_{32} & G_{33} \end{bmatrix},$$

where G_{23}, G_{32}, G_{33} are arbitrary.
(ii): Arbitrary $B^- \in \{B^-\}$ has a matrix form:

$$B^- = \begin{bmatrix} C_1^{-1} & 0 & F_{13} \\ 0 & C_2^{-1} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}.$$

The operator

$$A^- = \begin{bmatrix} C_1^{-1} & 0 & F_{13} \\ 0 & 0 & 0 \\ F_{31} & 0 & G_{33} \end{bmatrix},$$

where G_{33} is arbitrary, has desired properties. □

As we know from Theorem 3.7 (i) $\Leftrightarrow$ (viii), $A <^- B$ if and only if there exist projections P and Q such that $A = PB = BQ$. We obtain the class of all such projections. The following theorems are analogous to Theorems 3.5.13 - 3.5.18 in [20]. All of them can be proved using operators in matrix form with respect to standard decomposition.

Theorem 3.11. *Let $A, B \in \mathcal{B}_{reg}(X, Y)$ such that $A <^- B$. Then the class of all projections $P \in \mathcal{B}(Y)$ such that $A = PB$ is given by*

$$P = \begin{bmatrix} I & 0 & V(I - P_{33}) \\ 0 & 0 & UP_{33} \\ 0 & 0 & P_{33} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B - A) \\ Y_1 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B - A) \\ Y_1 \end{bmatrix},$$

where $P_{33} \in \mathcal{B}(Y_1)$ is some projection, and $U \in \mathcal{B}(Y_1, \mathcal{R}(B - A))$, $V \in \mathcal{B}(Y_1, \mathcal{R}(A))$ are arbitrary operators.

Proof. If P has the given form, then P is a projection and $A = PB$. Let P be a projection such that $A = PB$. Suppose that $P = [P_{ij}]$, $i, j \in \{1, 2, 3\}$, with respect to standard decomposition. From $A = PB$ we conclude that $P_{11} = I, P_{12} = P_{21} = P_{22} = P_{31} = P_{32} = 0$ and from $P^2 = P$ we conclude that $P_{23} = P_{23}P_{33}$, $P_{13} = P_{13} + P_{13}P_{33}$ and $P_{33} = P_{33}^2$. Hence $P_{13} = V(I - P_{33})$ and $P_{23} = UP_{33}$ where U and V are arbitrary. □

In the same manner we obtain the following theorem.

Theorem 3.12. *Let $A, B \in \mathcal{B}_{reg}(X, Y)$ such that $A <^- B$. Then the class of all projections $Q \in \mathcal{B}(X)$ such that $A = BQ$ is given by*

$$Q = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ (I - Q_{33})V & Q_{33}U & Q_{33} \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \\ \mathcal{N}(B) \end{bmatrix},$$

where $Q_{33} \in \mathcal{B}(\mathcal{N}(B))$ is some projection and $U \in \mathcal{B}(X_2, \mathcal{N}(B))$, $V \in \mathcal{B}(X_1, \mathcal{N}(B))$ are arbitrary operators.

Remark 3.13. Theorem 3.11 yields that $Y = \mathcal{R}(A) \oplus \mathcal{R}(B - A) \oplus \mathcal{R}(P_{33}) \oplus \mathcal{N}(P_{33})$. Of course, $X = X_1 \oplus X_2 \oplus \mathcal{N}(B)$ has standard decomposition. With respect to this decompositions, we obtain that A, B and P have the following forms:

$$A = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ P = \begin{bmatrix} I & 0 & 0 & V \\ 0 & 0 & U & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

for some operators U and V.
Similarly, Theorem 3.12 yields that $X_1 \oplus X_2 \oplus \mathcal{R}(Q_{33}) \oplus \mathcal{N}(Q_{33})$. With respect to this decomposition A, B and Q have the following forms:

$$A = \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ Q = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & L & I & 0 \\ M & 0 & 0 & 0 \end{bmatrix},$$

for some operators L and M.

Theorem 3.14. *Let $A, B \in \mathcal{B}_{reg}(X, Y)$ such that $A <^- B$. Then the class of all projections P such that $A = PB$ and $\mathcal{R}(P) = \mathcal{R}(A)$ is given by $\{AA^- : A^- \in \{A^-\}_B\}$. The class of all projections Q such that $A = BQ$ and $\mathcal{N}(Q) = \mathcal{N}(A)$ is given by $\{A^-A : A^- \in \{A^-\}_B\}$.*

Proof. According to Theorems 3.11 and 3.12 we see that the class of all projections P such that $A = PB$, $\mathcal{R}(P) = \mathcal{R}(A)$ and all projections Q such that $A = BQ$, $\mathcal{N}(Q) = \mathcal{N}(A)$ have the following forms:

$$P = \begin{bmatrix} I & 0 & V_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ V_2 & 0 & 0 \end{bmatrix},$$

respectively, where V_1 and V_2 are arbitrary. From the proof of Theorem 3.8 (i) we see that AA^- and A^-A where $A^- \in \{A^-\}_B$ have the above forms, respectively. □

Theorem 3.15. *Let $A, B \in \mathcal{B}_{reg}(X, Y)$ such that $A <^- B$. If P is the projection such that $A = PB$, then P can be written as $P = P_1 + P_2$, where P_1 is a projection such that $A = P_1B$, $\mathcal{R}(P_1) = \mathcal{R}(A)$, and P_2 is a projection such that $P_1P_2 = P_2P_1 = P_2A = P_2B = 0$. If Q is the projection such that $A = BQ$, then Q can be written as $Q = Q_1 + Q_2$, where Q_1 is a projection such that $A = BQ_1$, $\mathcal{N}(Q_1) = \mathcal{N}(A)$, and Q_2 is a projection such that $Q_1Q_2 = Q_2Q_1 = AQ_2 = BQ_2 = 0$.*

Proof. According to Theorems 3.11, 3.12 and 3.14 we can take

$$P_1 = \begin{bmatrix} I & 0 & V(I - P_{33}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & UP_{33} \\ 0 & 0 & P_{33} \end{bmatrix} \text{ and}$$

$$Q_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ (I - Q_{33})V & 0 & 0 \end{bmatrix}, \ Q_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & Q_{33}U & Q_{33} \end{bmatrix}.$$

It is easy to show that P_1, P_2 and Q_1, Q_2 satisfy the conditions of the theorem. □

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